

NLO structure functions at small-x

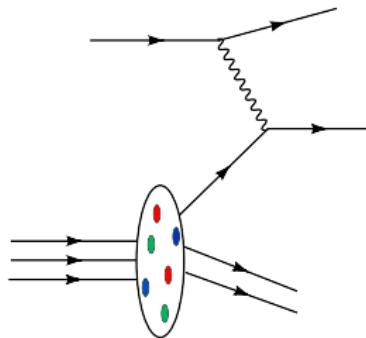
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Lawrence Berkeley Laboratory

DIS - Newport News, 10 - 15 April 2011

- Light-cone OPE versus OPE in color dipoles.
- High-energy scattering and Wilson lines formalism.
- Factorization in rapidity.
- NLO Photon Impact Factor: analytic result.
- Brief review of the LO and NLO BK equation.
- Conclusions and outlook.

Incoherent Interactions



Bjorken Limit

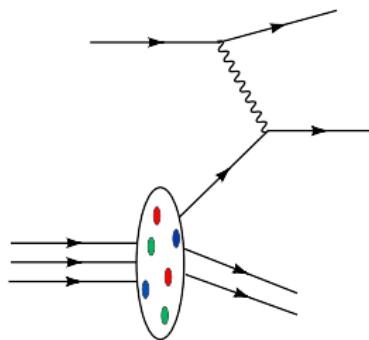
$$Q^2 \rightarrow \infty, s \rightarrow \infty$$

$$x_B = \frac{Q^2}{s} \text{ fixed}$$

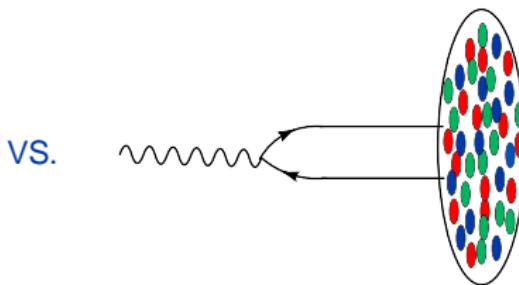
resum $\alpha_s \ln \frac{Q^2}{\Lambda_{\text{QCD}}}$

Incoherent-vs-Coherent

Incoherent Interactions



Coherent Interactions



Bjorken Limit

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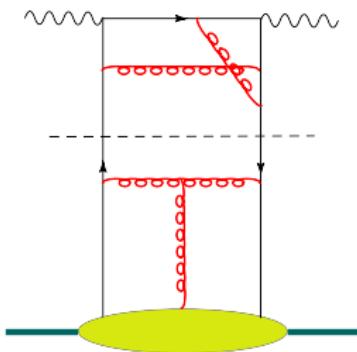
Regge Limit

$$Q^2 \text{ fixed, } s \rightarrow \infty$$

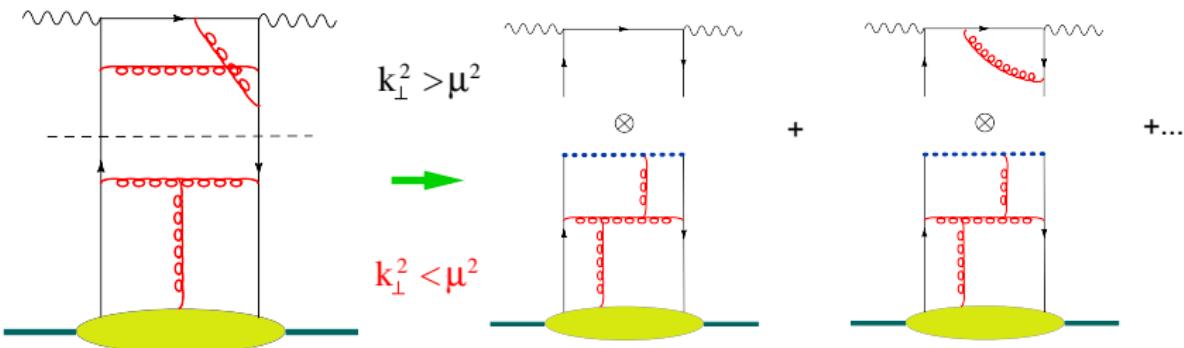
$$x_B = \frac{Q^2}{s} \rightarrow 0$$

$$\text{resum } \alpha_s \ln \frac{1}{x_B}$$

Light-cone expansion and DGLAP evolution in the NLO



Light-cone expansion and DGLAP evolution in the NLO

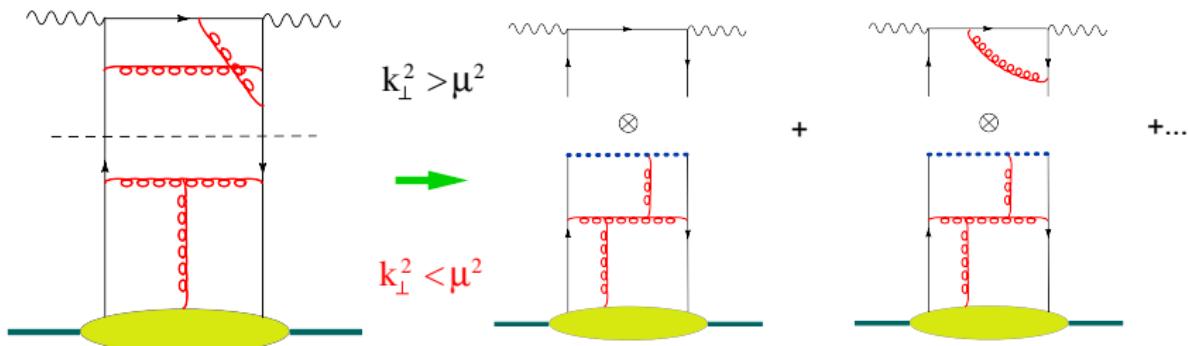


μ^2 - factorization scale (normalization point)

$k_{\perp}^2 > \mu^2$ - coefficient functions

$k_{\perp}^2 < \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)

Light-cone expansion and DGLAP evolution in the NLO



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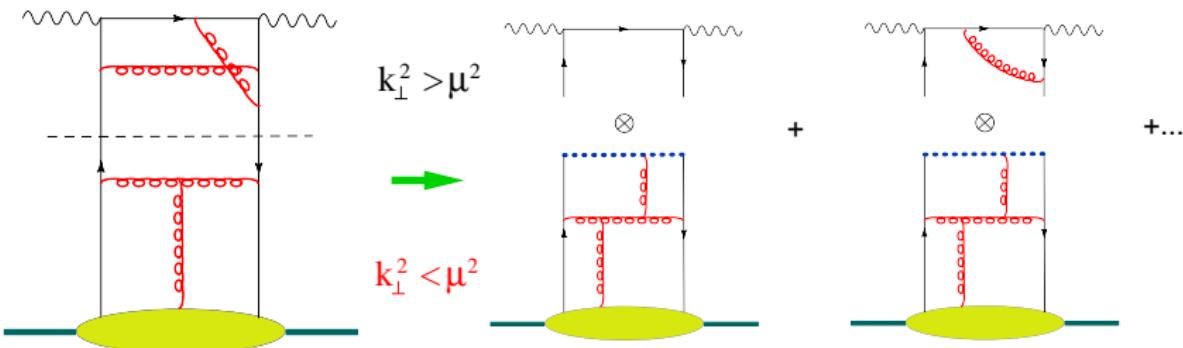
OPE in light-ray operators

$(x-y)^2 \rightarrow 0$

$$T\{j_\mu(x)j_\nu(y)\} = \frac{(x-y)_\xi}{2\pi^2(x-y)^4} \left[1 + \frac{\alpha_s}{\pi} (\ln(x-y)^2\mu^2 + C) \right] \bar{\psi}(x)\gamma_\mu\gamma^\xi\gamma_\nu[x,y]\psi(y)$$

$$[x,y] \equiv Pe^{ig\int_0^1 du (x-y)^\mu A_\mu(ux+(1-u)y)} - \text{gauge link}$$

Light-cone expansion and DGLAP evolution in the NLO



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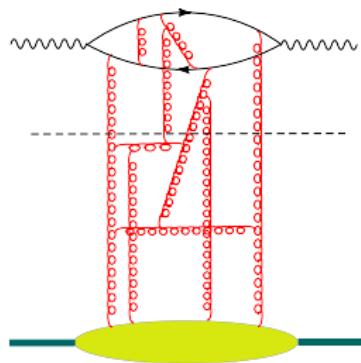
Renorm-group equation for light-ray operators \Rightarrow DGLAP evolution of

parton densities

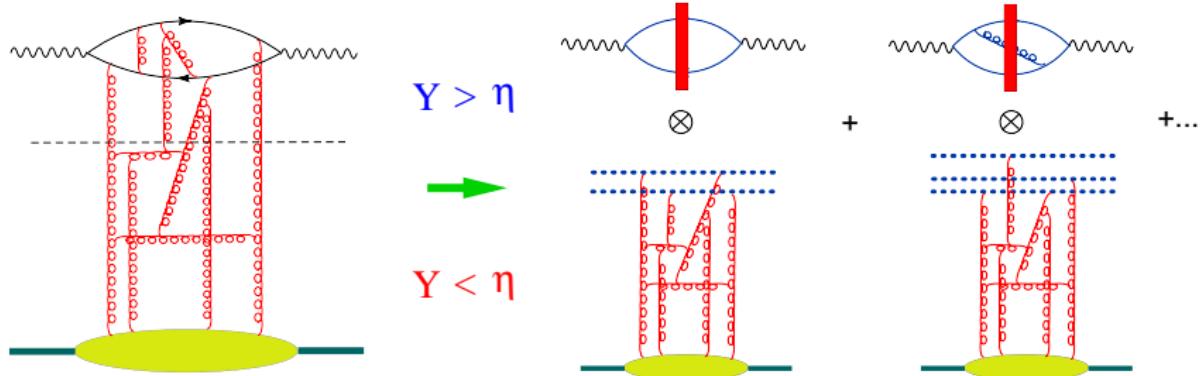
$$(x - y)^2 = 0$$

$$\mu^2 \frac{d}{d\mu^2} \bar{\psi}(x)[x, y]\psi(y) = K_{\text{LO}} \bar{\psi}(x)[x, y]\psi(y) + \alpha_s K_{\text{NLO}} \bar{\psi}(x)[x, y]\psi(y)$$

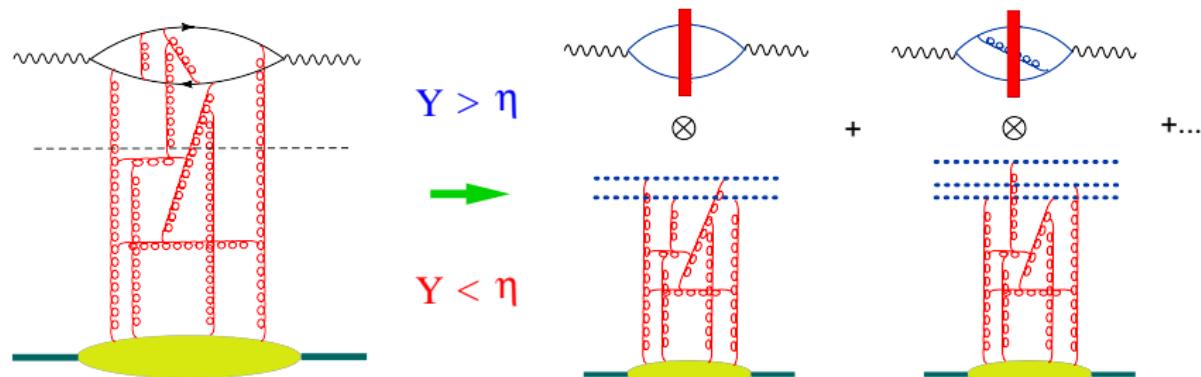
High-energy expansion in color dipoles in the NLO



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η - rapidity factorization scale

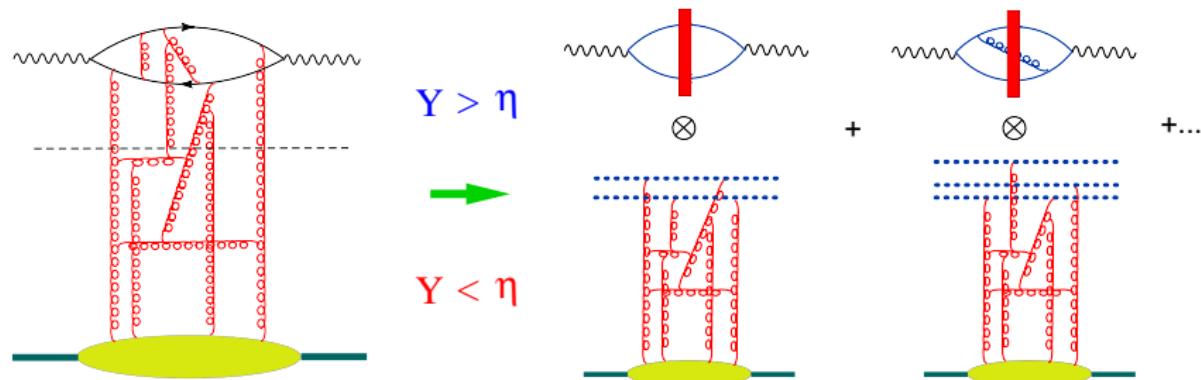
Rapidity $Y > \eta$ - coefficient function (“impact factor”)

Rapidity $Y < \eta$ - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta = \text{Pexp} \left[ig \int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp) \right]$$

$$A_\mu^\eta(x) = \int \frac{d^4 k}{(2\pi)^4} \theta(e^\eta - |\alpha_k|) e^{-ik \cdot x} A_\mu(k)$$

High-energy expansion in color dipoles in the NLO



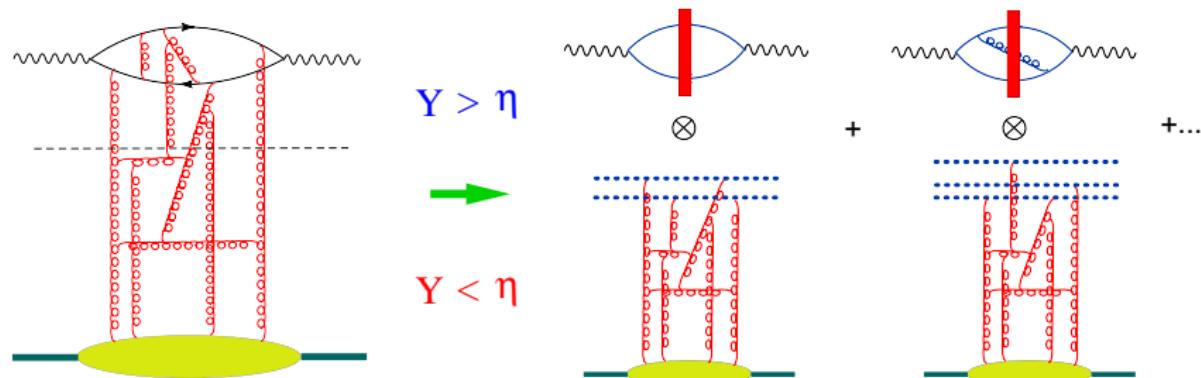
The high-energy operator expansion is

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 I_{\mu\nu}^{\text{LO}}(z_1, z_2, x, y) \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}$$
$$+ \int d^2z_1 d^2z_2 d^2z_3 I_{\mu\nu}^{\text{NLO}}(z_1, z_2, z_3, x, y) [\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]$$

In the leading order the impact factor is Möbius invariant

In the NLO one should also expect conf. invariance since $I_{\mu\nu}^{\text{NLO}}$ is given by tree diagrams

High-energy expansion in color dipoles in the NLO



η - rapidity factorization scale

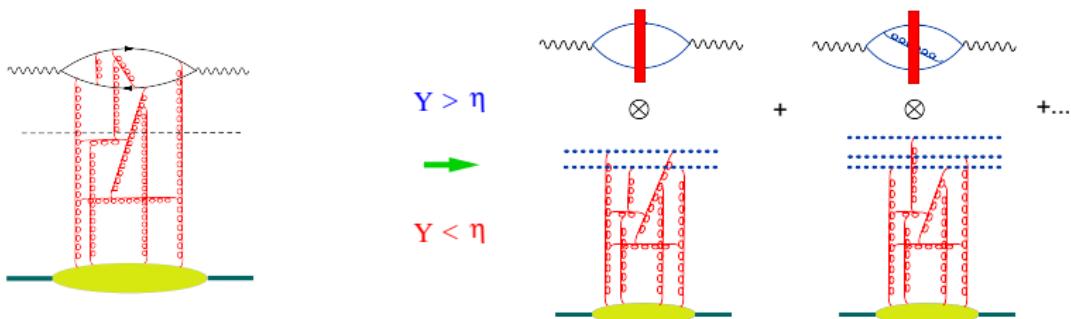
Evolution equation for color dipoles

$$\begin{aligned} \frac{d}{d\eta} \text{tr}\{U_x^\eta U_y^{\dagger\eta}\} &= \frac{\alpha_s}{2\pi^2} \int d^2 z \frac{(x-y)^2}{(x-z)^2(y-z)^2} [\text{tr}\{U_x^\eta U_y^{\dagger\eta}\} \text{tr}\{U_x^\eta U_y^{\dagger\eta}\} \\ &\quad - N_c \text{tr}\{U_x^\eta U_y^{\dagger\eta}\}] + \alpha_s K_{\text{NLO}} \text{tr}\{U_x^\eta U_y^{\dagger\eta}\} + O(\alpha_s^2) \end{aligned}$$

$$K_{\text{NLO}}=?$$

(Linear part of $K_{\text{NLO}} = K_{\text{NLO BFKL}}$)

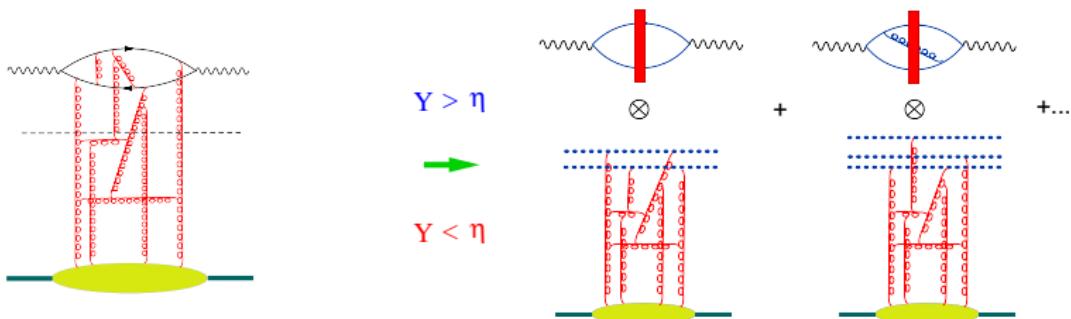
Expansion of $F_2(x)$ in color dipoles in the next-to-leading order



$$F_2(x_B) \simeq \int d^2 z_1 d^2 z_2 I^{LO}(z_1, z_2) \langle \text{tr}\{U_{z_1}^\eta U_{z_2}^{\dagger\eta}\} \rangle \quad \eta = \ln \frac{1}{x_B}$$

$$+ \frac{\alpha_s}{\pi} \int d^2 z_1 d^2 z_2 d^2 z_3 I^{NLO}(z_1, z_2, z_3) \langle \text{tr}\{U_{z_1}^\eta U_{z_3}^{\dagger\eta}\} \text{tr}\{U_{z_3} U_{z_2}^{\dagger\eta}\} \rangle$$

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plan

- Calculate the NLO photon impact factor.
- Calculate the NLO evolution of color dipole.
- Convolute the solution with the initial conditions for the evolution and get the amplitude.

Propagation in the shock wave: Wilson line (Spectator frame)



Each path is weighted with the gauge factor $P e^{ig \int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction \Rightarrow we can replace the gauge factor along the actual path with the one along the straight-line path.



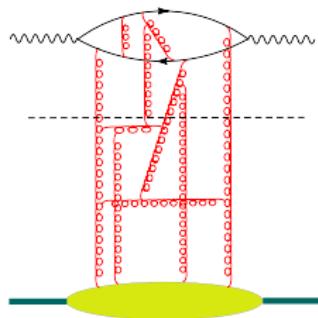
$$U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp]$$

$$[x, y] = P e^{ig \int_0^1 du (x-y)^\mu A_\mu (u x + (1-u)y)} \quad p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu$$

Propagation in the shock wave: Wilson line (Spectator frame)



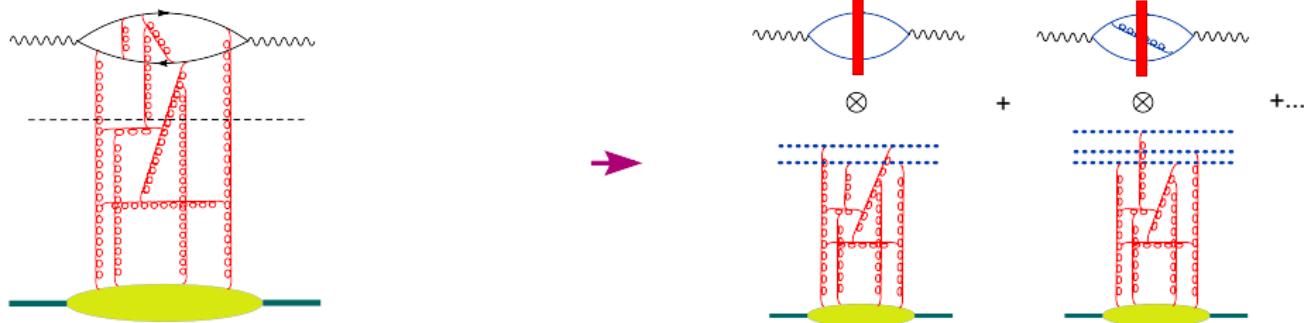
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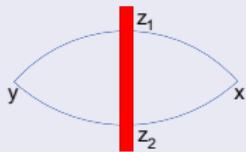
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LO and NLO Impact Factor

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 I_{\mu\nu}^{\text{LO}}(z_1, z_2, x, y) \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}$$
$$+ \int d^2z_1 d^2z_2 d^2z_3 I_{\mu\nu}^{\text{NLO}}(z_1, z_2, z_3, x, y) [\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]$$

LO Impact Factor diagram: I^{LO}



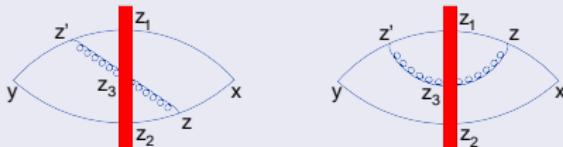
LO and NLO Impact Factor

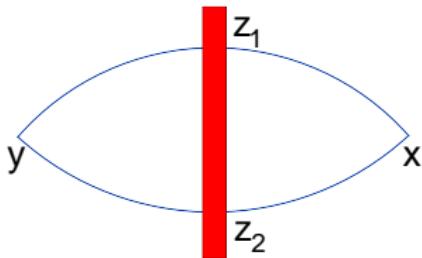
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LO Impact Factor diagram: I^{LO}



NLO Impact Factor diagrams: I^{NLO}





Conformal vectors:

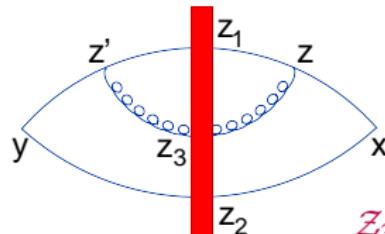
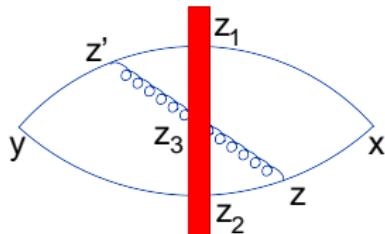
$$\kappa = \frac{\sqrt{s}}{2x_*} \left(\frac{p_1}{s} - x_\perp^2 p_2 + x_\perp \right) - \frac{\sqrt{s}}{2y_*} \left(\frac{p_1}{s} - y_\perp^2 p_2 + y_\perp \right)$$

$$\zeta_1 = \left(\frac{p_1}{s} + z_{1\perp}^2 p_2 + z_{1\perp} \right), \quad \zeta_2 = \left(\frac{p_1}{s} + z_{2\perp}^2 p_2 + z_{2\perp} \right)$$

Here $x^2 = -x_\perp^2$, $x_* \equiv x_\mu p_2^\mu$ (similarly for y); $\mathcal{R} = \frac{\kappa^2 (\zeta_1 \cdot \zeta_2)}{2(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2)}$

$$I_{\mu\nu}^{\text{LO}}(z_1, z_2) = \frac{\mathcal{R}^2}{\pi^6 (\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2)} \frac{\partial^2}{\partial x^\mu \partial y^\nu} [(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2) - \frac{1}{2}\kappa^2 (\zeta_1 \cdot \zeta_2)]$$

NLO Impact Factor

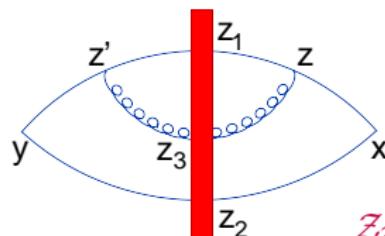
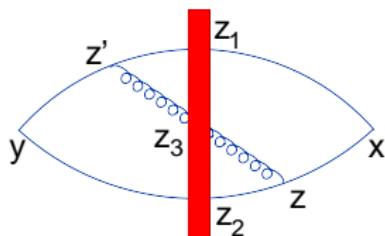


$$\mathcal{Z}_3 \equiv \frac{(x-z_3)_\perp^2}{x^+} - \frac{(y-z_3)_\perp^2}{y^+}$$

$$I_{\mu\nu}^{\text{NLO}}(x, y; z_1, z_2, z_3; \eta) = - I_{\mu\nu}^{\text{LO}} \times \frac{\alpha_s}{2\pi} \frac{z_{13}^2}{z_{12}^2 z_{23}^2} \ln \frac{\sigma s}{4} \mathcal{Z}_3 + \text{conf.}$$

The NLO impact factor is not Möbius invariant \Rightarrow the color dipole with the cutoff $\eta = \ln \sigma$ is not invariant.

NLO Impact Factor



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However, if we define a composite operator (a - analog of μ^{-2} for usual OPE)

$$\begin{aligned} [\text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} &= \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\ &+ \frac{\alpha_s}{4\pi} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[\frac{1}{N_c} \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \right] \ln \frac{az_{12}^2}{z_{13}^2 z_{23}^2} + O(\alpha_s^2) \end{aligned}$$

the impact factor becomes conformal in the NLO.

Operator expansion in conformal dipoles

Analogy:

When the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator must be corrected by finite counter-terms order by order in perturbation theory.

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 I_{\mu\nu}^{\text{LO}}(z_1, z_2, x, y) \text{tr}[\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}}$$
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The new NLO impact factor is conformally invariant.

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The new NLO impact factor is conformally invariant.

In conformal $\mathcal{N} = 4$ SYM theory one can construct the composite conformal dipole operator order by order in perturbation theory.

$$\Delta \equiv (x - y), \quad x_* = x^+ \sqrt{s/2}, \quad y_* = y^+ \sqrt{s/2}, \quad R \equiv \frac{\Delta^2 z_{12}^2}{x_* y_* \mathcal{Z}_1 \mathcal{Z}_2}$$

$$\begin{aligned}
I_{\mu\nu}^{NLO}(x, y) = & \frac{\alpha_s}{4\pi^7 \Delta^4} \frac{\partial \kappa^\alpha}{\partial x^\mu} \frac{\partial \kappa^\beta}{\partial y^\nu} \int \frac{dz_1 dz_2}{z_{12}^4} \mathcal{U}(z_1, z_2) R^2 \left\{ -\frac{2}{\kappa^2} \left(g^{\alpha\beta} - 2 \frac{\kappa^\alpha \kappa^\beta}{\kappa^2} \right) \right. \\
& + \frac{\zeta_1^\alpha \zeta_2^\beta + \zeta_1 \leftrightarrow \zeta_2}{(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2)} \left[4\text{Li}_2(1-R) - \frac{2\pi^2}{3} + \frac{2\ln R}{1-R} + \frac{\ln R}{R} - 4\ln R + \frac{1}{2R} - 2 - 4C - \frac{2C}{R} \right. \\
& + 2(\ln \frac{1}{R} + \frac{1}{R} - 2) \left(\ln \frac{1}{R} + 2C \right) \left. \right] + \left(\frac{\zeta_1^\alpha \zeta_1^\beta}{(\kappa \cdot \zeta_1)^2} + \zeta_1 \leftrightarrow \zeta_2 \right) \left[\frac{\ln R}{R} - \frac{2C}{R} + 2\frac{\ln R}{1-R} - \frac{1}{2R} \right] \\
& + \left[-2\frac{\ln R}{1-R} - \frac{\ln R}{R} + \ln R - \frac{3}{2R} + \frac{5}{2} + 2C + \frac{2C}{R} \right] \left[\frac{\zeta_1^\alpha \kappa^\beta + \zeta_1^\beta \kappa^\alpha}{(\kappa \cdot \zeta_1)\kappa^2} + \zeta_1 \leftrightarrow \zeta_2 \right] \\
& + \frac{g^{\alpha\beta}(\zeta_1 \cdot \zeta_2)}{(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2)} \left[\frac{2\pi^2}{3} - 4\text{Li}_2(1-R) - 2 \left(\ln \frac{1}{R} + \frac{1}{R} + \frac{1}{2R^2} - 3 \right) \left(\ln \frac{1}{R} + 2C \right) \right. \\
& \left. \left. + 6\ln R - \frac{2}{R} + 2 + \frac{3}{2R^2} \right] \right\}
\end{aligned}$$

$$\Delta \equiv (x - y), \quad x_* = x^+ \sqrt{s/2}, \quad y_* = y^+ \sqrt{s/2}, \quad R \equiv \frac{\Delta^2 z_{12}^2}{x_* y_* \bar{z}_1 \bar{z}_2}$$

$$\begin{aligned}
I_{\mu\nu}^{NLO}(x, y) = & \frac{\alpha_s}{4\pi^7 \Delta^4} \frac{\partial \kappa^\alpha}{\partial x^\mu} \frac{\partial \kappa^\beta}{\partial y^\nu} \int \frac{dz_1 dz_2}{z_{12}^4} \mathcal{U}(z_1, z_2) R^2 \left\{ -\frac{2}{\kappa^2} \left(g^{\alpha\beta} - 2 \frac{\kappa^\alpha \kappa^\beta}{\kappa^2} \right) \right. \\
& + \frac{\zeta_1^\alpha \zeta_2^\beta + \zeta_1 \leftrightarrow \zeta_2}{(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2)} \left[4\text{Li}_2(1-R) - \frac{2\pi^2}{3} + \frac{2\ln R}{1-R} + \frac{\ln R}{R} - 4\ln R + \frac{1}{2R} - 2 - 4C - \frac{2C}{R} \right. \\
& + 2(\ln \frac{1}{R} + \frac{1}{R} - 2) \left(\ln \frac{1}{R} + 2C \right) \left. \right] + \left(\frac{\zeta_1^\alpha \zeta_1^\beta}{(\kappa \cdot \zeta_1)^2} + \zeta_1 \leftrightarrow \zeta_2 \right) \left[\frac{\ln R}{R} - \frac{2C}{R} + 2\frac{\ln R}{1-R} - \frac{1}{2R} \right] \\
& + \left[-2\frac{\ln R}{1-R} - \frac{\ln R}{R} + \ln R - \frac{3}{2R} + \frac{5}{2} + 2C + \frac{2C}{R} \right] \left[\frac{\zeta_1^\alpha \kappa^\beta + \zeta_1^\beta \kappa^\alpha}{(\kappa \cdot \zeta_1)\kappa^2} + \zeta_1 \leftrightarrow \zeta_2 \right] \\
& + \frac{g^{\alpha\beta}(\zeta_1 \cdot \zeta_2)}{(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2)} \left[\frac{2\pi^2}{3} - 4\text{Li}_2(1-R) - 2 \left(\ln \frac{1}{R} + \frac{1}{R} + \frac{1}{2R^2} - 3 \right) \left(\ln \frac{1}{R} + 2C \right) \right. \\
& \left. \left. + 6\ln R - \frac{2}{R} + 2 + \frac{3}{2R^2} \right] \right\}
\end{aligned}$$

Photon Impact Factor at NLO

Conformal vectors

$$\begin{aligned}\kappa^\mu &= \frac{\sqrt{s}}{2x_*} \left(\frac{p_1^\mu}{s} - x^2 p_2^\mu + x_\perp^\mu \right) - \frac{\sqrt{s}}{2y_*} \left(\frac{p_1^\mu}{s} - y^2 p_2^\mu + y_\perp^\mu \right) \\ \zeta_1^\mu &= \left(\frac{p_1^\mu}{s} + z_{1\perp}^2 p_2^\mu + z_{1\perp}^\mu \right), \quad \zeta_2^\mu = \left(\frac{p_1^\mu}{s} + z_{2\perp}^2 p_2^\mu + z_{2\perp}^\mu \right)\end{aligned}$$

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DIS photon impact factor is a linear combination of the following tensor basis

$$\mathcal{I}_1^{\mu\nu} = g^{\mu\nu} \quad \mathcal{I}_2^{\mu\nu} = \frac{\kappa^\mu \kappa^\nu}{\kappa^2}$$

$$\mathcal{I}_3^{\mu\nu} = \frac{\kappa^\mu \zeta_1^\nu + \kappa^\nu \zeta_1^\mu}{\kappa \cdot \zeta_1} + \frac{\kappa^\mu \zeta_2^\nu + \kappa^\nu \zeta_2^\mu}{\kappa \cdot \zeta_2}$$

$$\mathcal{I}_4^{\mu\nu} = \frac{\kappa^2 \zeta_1^\mu \zeta_1^\nu}{(\kappa \cdot \zeta_1)^2} + \frac{\kappa^2 \zeta_2^\mu \zeta_2^\nu}{(\kappa \cdot \zeta_2)^2} \quad \mathcal{I}_5^{\mu\nu} = \frac{\zeta_1^\mu \zeta_2^\nu + \zeta_2^\mu \zeta_1^\nu}{\zeta_1 \cdot \zeta_2}$$

Cornalba, Costa, Penedones (2010)

Projection of the LO impact factor on the eigenfunctions

$$\int \frac{d^2 z_1 d^2 z_2}{z_{12}^2} I_{LO}^{\mu\nu}(z_1, z_2) \left(\frac{\kappa^2}{(\kappa \cdot \zeta_0)^2} \right)^\gamma = B(1-\gamma)\Gamma(\gamma+2)\Gamma(3-\gamma)$$
$$\times \left\{ \frac{\gamma(1-\gamma)D_1}{12(1+\gamma)(2-\gamma)} + \frac{D_2}{2(1+\gamma)(2-\gamma)} - \frac{D_3^{\mu\nu}}{8(1+\gamma)(2-\gamma)} \right.$$
$$- \frac{\gamma(1-\gamma)D_4^{\mu\nu}}{16(1+2\gamma)(3-2\gamma)(1+\gamma)(2-\gamma)} - \frac{D_1^{\mu\nu} + D^{\mu\nu}\nu_2}{8} \left. \right\}_{\mu\nu} \left(\frac{\kappa^2}{(\kappa \cdot \zeta_0)^2} \right)^\gamma$$

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where

$$(D_1 + D_2)^{\mu\nu} = -2\Delta^2 x_* y_* \kappa^{-2} \partial_x^\mu \partial_y^\nu \kappa^2$$

$$D_2^{\mu\nu} = -\Delta^2 x_* y_* \partial_x^\mu (\ln \kappa^2) \partial_y^\nu \ln \kappa^2$$

$$D_3^{\mu\nu} = 4\gamma \Delta^2 x_* y_* [(\partial_x^\mu \ln \kappa^2) \partial_y^\nu \ln(\kappa \cdot \zeta_0) + (\partial_y^\nu \ln \kappa^2) \partial_x^\mu \ln(\kappa \cdot \zeta_0) - (\partial_x^\mu \ln \kappa^2) \partial_y^\nu \ln \kappa^2]$$

$$D_4^{\mu\nu} = 4\gamma(1+2\gamma) \Delta^2 x_* y_* \left[-\frac{1}{3} \partial_x^\mu \partial_y^\nu \ln \kappa^2 - \partial_x^\mu (\ln \kappa^2) \partial_y^\nu \ln \kappa^2 \right.$$

$$\left. + (\partial_x^\mu \ln \kappa^2) \partial_y^\nu \ln(\kappa \cdot \zeta_0) + (\partial_y^\nu \ln \kappa^2) \partial_x^\mu \ln(\kappa \cdot \zeta_0) - 2\partial_\mu^\nu \ln(\kappa \cdot \zeta_0) \partial_y^\nu \ln(\kappa \cdot \zeta_0) \right]$$

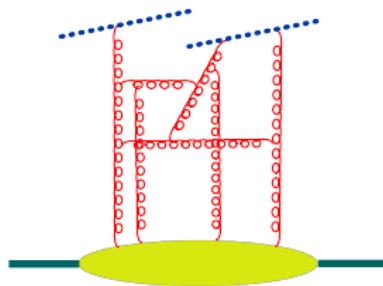
$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, $C = -\psi(1)$ is the Euler constant, and $\psi'(a) = \frac{d}{da} \ln \Gamma(a)$

Projection of the NLO impact factor on the eigenfunctions

$$\begin{aligned}
& \int \frac{d^2 z_1 d^2 z_2}{z_{12}^4} I_{NLO}^{\mu\nu}(z_1, z_2) \left(\frac{\kappa^2}{(\kappa \cdot \zeta_0)^2} \right)^\gamma = \alpha_s \frac{B(1-\gamma)\Gamma(3-\gamma)\Gamma(2+\gamma)}{(2-\gamma)(1+\gamma)} \times \\
& \left\{ \frac{D_1^{\mu\nu}}{3 \sin^2(\gamma\pi)} \left[(1 - \cos(2\gamma\pi)) \left(\chi - \gamma(1-\gamma) \left(C\chi - \frac{1}{2} \right) \right) - 1 - \gamma(1-\gamma) \frac{\pi^2}{3} (5 + \cos(2\gamma\pi)) \right] \right. \\
& + D_2^{\mu\nu} \left[-\frac{3}{\gamma(1-\gamma)} + 2\chi \left(\frac{1}{\gamma(1-\gamma) - 2C + 1} \right) + \frac{4}{3}\pi^2 \left(1 - \frac{3}{\sin^2(\gamma\pi)} \right) \right] \\
& + D_3^{\mu\nu} \left[C\chi - \frac{1}{2} - \frac{1}{\gamma(1-\gamma)} - \frac{\chi}{4} \left(1 - \frac{2}{\gamma(1-\gamma)} \right) - \frac{\pi^2}{3} \left(1 - \frac{3}{\sin^2(\gamma\pi)} \right) \right] \\
& + D_4^{\mu\nu} \left[\frac{15 + 10\gamma - 9\gamma^2 - \gamma^3(2-\gamma)}{4\gamma[3 + \gamma(1 - 4\gamma(2-\gamma))] } - \frac{1}{2(3 + 4\gamma(1-\gamma))} \left(\chi + \gamma(1-\gamma)(C\chi - \frac{\pi^2}{3} + \frac{\pi^2}{\sin^2(\gamma\pi)}) \right) \right] \\
& + \frac{1}{4} (D_1^{\mu\nu} + D_2^{\mu\nu}) (2-\gamma)(1+\gamma) \left[4\psi'(1-\gamma) - 4\psi'(3) + 4C\chi(\gamma) + 2\psi'(3-\gamma) + 2\psi'(2+\gamma) - 4\psi'(3) \right. \\
& + [\psi(3-\gamma) + \psi(2+\gamma) - 2\psi(1)]^2 - 6 - \frac{8}{(1+\gamma)(2-\gamma)} [\psi(2-\gamma) + \psi(1+\gamma) - 2\psi(1)] \\
& \left. - \frac{4}{\gamma(1-\gamma)(1+\gamma)(2-\gamma)} \left(-\chi(\gamma) + \frac{3}{2} \right) - 2\chi'(\gamma) - 2\chi^2(\gamma) \right] \left. \right\}
\end{aligned}$$

Regularization of the rapidity divergence

Matrix elements of Wilson lines: $\langle \text{Tr}\{U(x)U^\dagger(y)\} \rangle_A$ are divergent



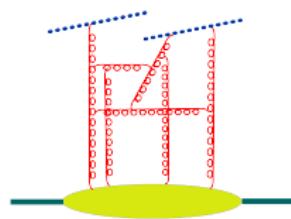
For light-like Wilson lines loop integrals are divergent in the longitudinal direction

$$\int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^\infty d\eta = \infty$$

$$F_2(x_B) \simeq \int d^2 z_1 d^2 z_2 I^{LO}(z_1, z_2) \langle \text{tr}\{U_{z_1}^\eta U_{z_2}^{\dagger \eta}\} \rangle \quad \eta = \ln \frac{1}{x_B}$$
$$+ \frac{\alpha_s}{\pi} \int d^2 z_1 d^2 z_2 d^2 z_3 I^{NLO}(z_1, z_2, z_3) \langle \text{tr}\{U_{z_1}^\eta U_{z_3}^{\dagger \eta}\} \text{tr}\{U_{z_3} U_{z_2}^{\dagger \eta}\} \rangle$$

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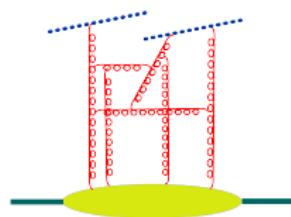
$$\int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^\infty d\eta = \infty$$

Regularization by: slope

$$U^\eta(x_\perp) = \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} du n_\mu A^\mu (un + x_\perp) \right\} \quad n^\mu = p_1^\mu + e^{-2\eta} p_2^\mu$$

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Regularization by: Rigid cut-off (used in NLO)

$$U_x^\eta = \text{Pexp} \left[ig \int_{-\infty}^\infty du p_1^\mu A_\mu^\eta (up_1 + x_\perp) \right]$$
$$A_\mu^\eta(x) = \int \frac{d^4 k}{(2\pi)^4} \theta(e^\eta - |\alpha_k|) e^{-ik \cdot x} A_\mu(k)$$

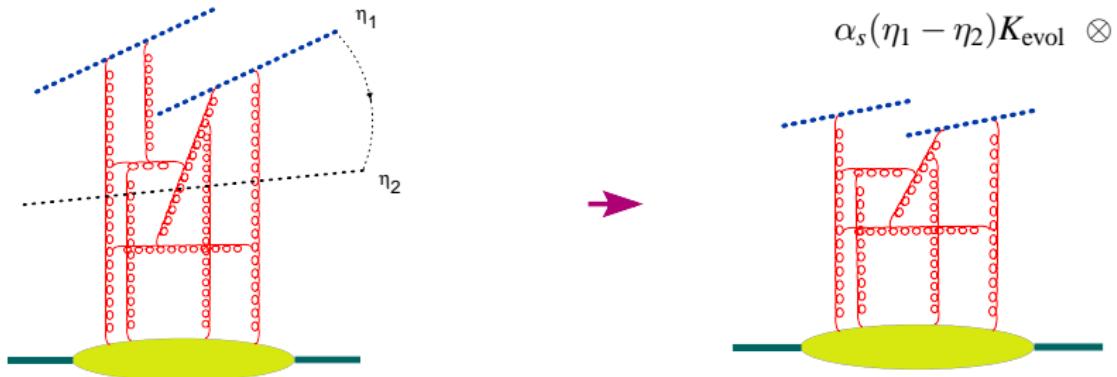
$$k^\mu = \alpha_k p_1^\mu + \beta_k p_2^\mu + k_\perp^\mu$$

Evolution Equation

$$\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \Rightarrow \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle$$

To get the evolution equation, consider the dipole with the rapidities up to η_1 and integrate over the gluons with rapidity $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidity up to η_2).

In the frame || to η_1 the gluons with $\eta < \eta_1$ are seen as pancake.

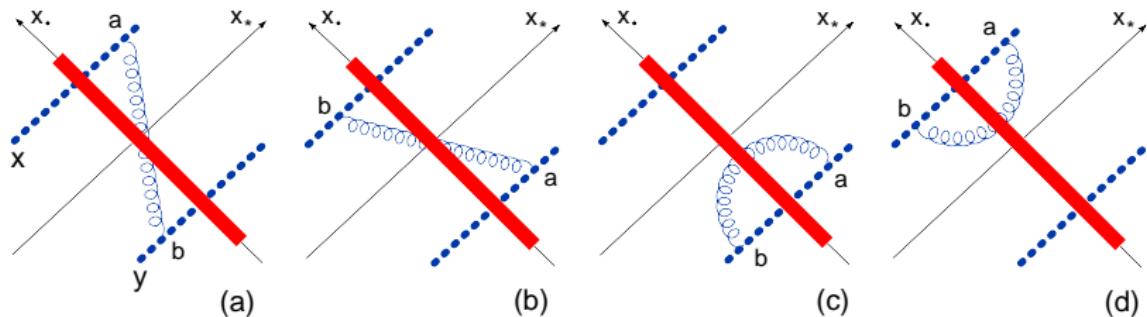


Particles with different rapidity perceive each other as Wilson lines.

Leading order: BK equation

$$\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \dots \Rightarrow$$

$$\frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}} = \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}}$$



$$x_\bullet = \sqrt{\frac{s}{2}} x^-$$

$$x_* = \sqrt{\frac{s}{2}} x^+$$

Non-linear evolution equation: BK equation

$$U_z^{ab} = \text{Tr}\{t^a U_z t^b U_z^\dagger\} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

Non-linear evolution equation: BK equation

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$$\hat{\mathcal{U}}(x, y) \equiv 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}(x_\perp) \hat{U}^\dagger(y_\perp)\}$$

BK equation: **Ian Balitsky (1996), Yu. Kovchegov (1999)**

$$\frac{d}{d\eta} \hat{\mathcal{U}}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 z (x-y)^2}{(x-z)^2 (y-z)^2} \left\{ \hat{\mathcal{U}}(x, z) + \hat{\mathcal{U}}(z, y) - \hat{\mathcal{U}}(x, y) - \hat{\mathcal{U}}(x, z) \hat{\mathcal{U}}(z, y) \right\}$$

Alternative approach: JIMWLK (1997-2000)

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LLA for DIS in pQCD \Rightarrow BFKL

(LLA: $\alpha_s \ll 1, \alpha_s \eta \sim 1$)

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Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD \Rightarrow BFKL

(LLA: $\alpha_s \ll 1$, $\alpha_s \eta \sim 1$)

LLA for DIS in sQCD \Rightarrow BK eqn

(LLA: $\alpha_s \ll 1$, $\alpha_s \eta \sim 1$, $\alpha_s^2 A^{1/3} \sim 1$)

(s for semi-classical)

Conformal invariance of the BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] = \text{Pexp} \left\{ ig \int_{-\infty}^{\infty} dx^+ A_+(x^+, x_\perp) \right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

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\Rightarrow The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp/x_\perp^2)$

$$\frac{d}{d\eta} \text{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2} \int \frac{d^2 z}{z^4} \frac{(x-y)^2}{(x-z)^2(z-y)^2} [\text{Tr}\{V_x V_z^\dagger\} \text{Tr}\{V_z V_y^\dagger\} - N_c \text{Tr}\{V_x V_y^\dagger\}]$$

Conformal invariance of the BK equation

SL(2,C) for Wilson lines

$$\hat{S}_- \equiv \frac{i}{2}(K^1 + iK^2), \quad \hat{S}_0 \equiv \frac{i}{2}(D + iM^{12}), \quad \hat{S}_+ \equiv \frac{i}{2}(P^1 - iP^2)$$

$$[\hat{S}_0, \hat{S}_\pm] = \pm \hat{S}_\pm, \quad \frac{1}{2}[\hat{S}_+, \hat{S}_-] = \hat{S}_0,$$

$$[\hat{S}_-, \hat{U}(z, \bar{z})] = z^2 \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_0, \hat{U}(z, \bar{z})] = z \partial_z \hat{U}(z, \bar{z}), \quad [\hat{S}_+, \hat{U}(z, \bar{z})] = -\partial_z \hat{U}(z, \bar{z})$$

$$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 - iz^2, \quad U(z_\perp) = U(z, \bar{z})$$

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Conformal invariance of the evolution kernel

$$\begin{aligned} \frac{d}{d\eta} [\hat{S}_-, \text{Tr}\{U_x U_y^\dagger\}] &= \frac{\alpha_s N_c}{2\pi^2} \int dz K(x, y, z) [\hat{S}_-, \text{Tr}\{U_x U_y^\dagger\} \text{Tr}\{U_x U_y^\dagger\}] \\ &\Rightarrow \left[x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + z^2 \frac{\partial}{\partial z} \right] K(x, y, z) = 0 \end{aligned}$$

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$$z \equiv z^1 + iz^2, \bar{z} \equiv z^1 - iz^2, \quad U(z_\perp) = U(z, \bar{z})$$

Conformal invariance of the evolution kernel

$$\begin{aligned} \frac{d}{d\eta} [\hat{S}_-, \text{Tr}\{U_x U_y^\dagger\}] &= \frac{\alpha_s N_c}{2\pi^2} \int dz K(x, y, z) [\hat{S}_-, \text{Tr}\{U_x U_y^\dagger\} \text{Tr}\{U_x U_y^\dagger\}] \\ &\Rightarrow \left[x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + z^2 \frac{\partial}{\partial z} \right] K(x, y, z) = 0 \end{aligned}$$

In the leading order - OK. In the NLO - ?

Non-linear evolution equation at NLO

$$\begin{aligned} \frac{d}{d\eta} Tr\{U_x U_y^\dagger\} = & \\ & \int \frac{d^2 z}{2\pi^2} \left(\alpha_s \frac{(x-y)^2}{(x-z)^2(z-y)^2} + \alpha_s^2 K_{NLO}(x,y,z) \right) [Tr\{U_x U_z^\dagger\} Tr\{U_z U_y^\dagger\} - N_c Tr\{U_x U_y^\dagger\}] + \\ & \alpha_s^2 \int d^2 z d^2 z' \left(K_4(x,y,z,z') \{U_x, U_{z'}^\dagger, U_z, U_y^\dagger\} + K_6(x,y,z,z') \{U_x, U_{z'}^\dagger, U_{z'}, U_z, U_z^\dagger, U_y^\dagger\} \right) \end{aligned}$$

K_{NLO} is the next-to-leading order correction to the dipole kernel and K4 and K6 are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

Definition of the NLO kernel

In general

$$\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3)$$

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$$\alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} - \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3)$$

Definition of the NLO kernel

In general

$$\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3)$$

$$\alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} - \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3)$$

We calculate the “matrix element” of the r.h.s. in the shock-wave background

$$\langle \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle = \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle - \langle \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle + O(\alpha_s^3)$$

Definition of the NLO kernel

In general

$$\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3)$$

$$\alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = \frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} - \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + O(\alpha_s^3)$$

We calculate the “matrix element” of the r.h.s. in the shock-wave background

$$\langle \alpha_s^2 K_{\text{NLO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle = \frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle - \langle \alpha_s K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle + O(\alpha_s^3)$$

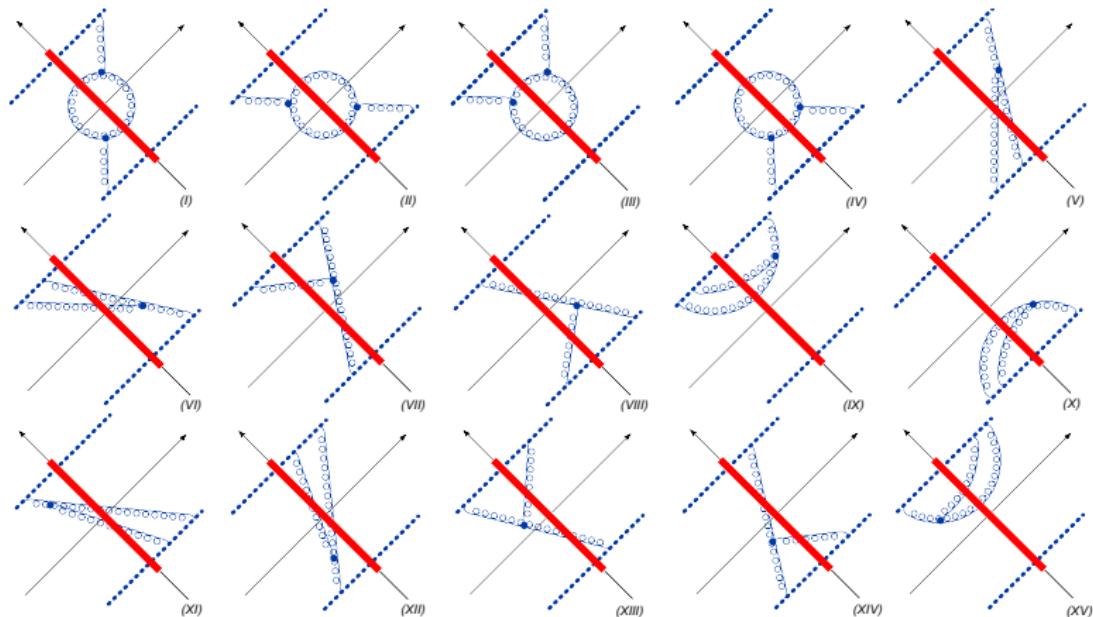
Subtraction of the (LO) contribution (with the rigid rapidity cutoff)
⇒ $\left[\frac{1}{v}\right]_+$ prescription in the integrals over Feynman parameter v

Typical integral

$$\int_0^1 dv \frac{1}{(k-p)_\perp^2 v + p_\perp^2 (1-v)} \left[\frac{1}{v}\right]_+ = \frac{1}{p_\perp^2} \ln \frac{(k-p)_\perp^2}{p_\perp^2}$$

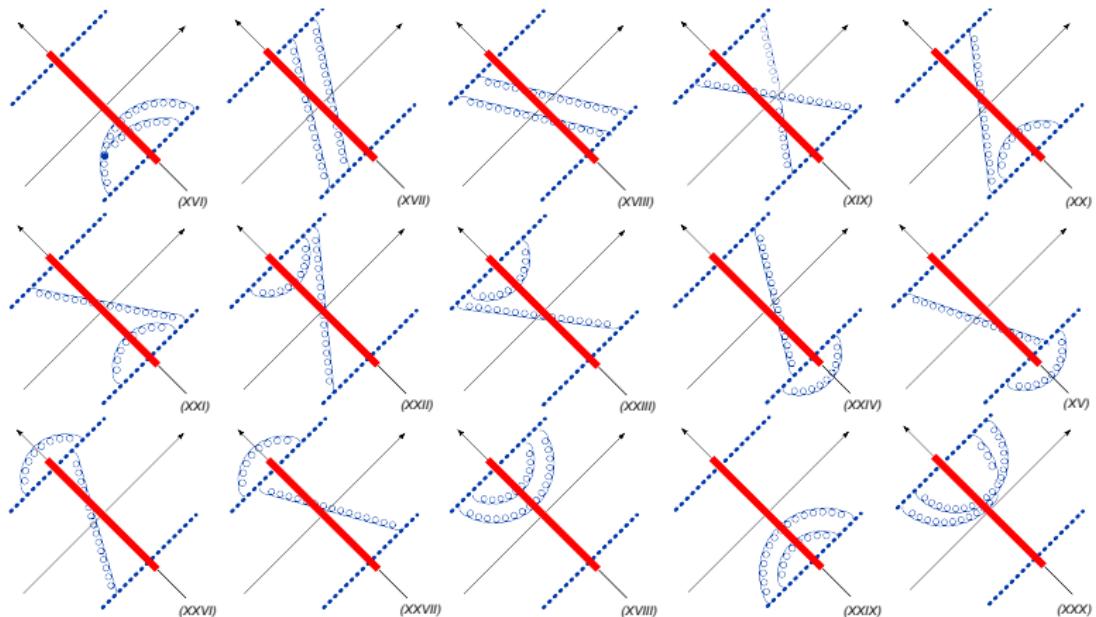
Diagrams of the NLO gluon contribution

Diagrams with 2 gluons interaction



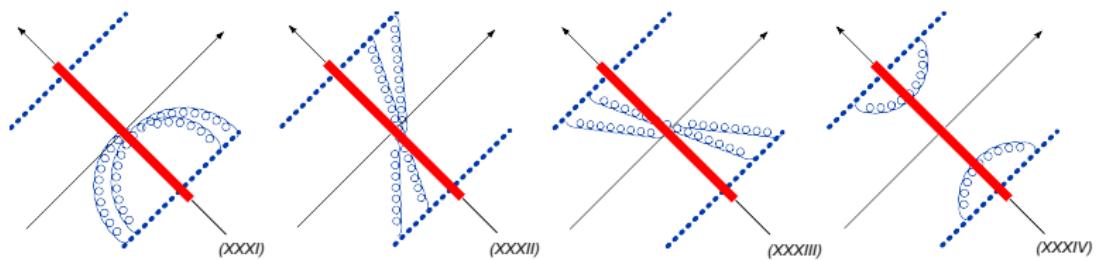
Diagrams of the NLO gluon contribution

Diagrams with 2 gluons interaction



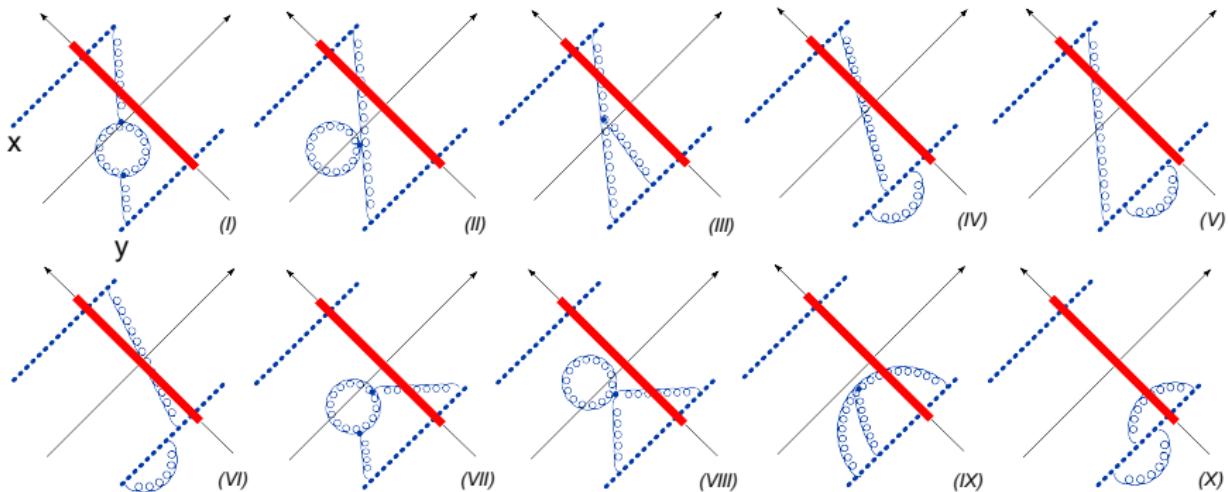
Diagrams of the NLO gluon contribution

Diagrams with 2 gluons interaction



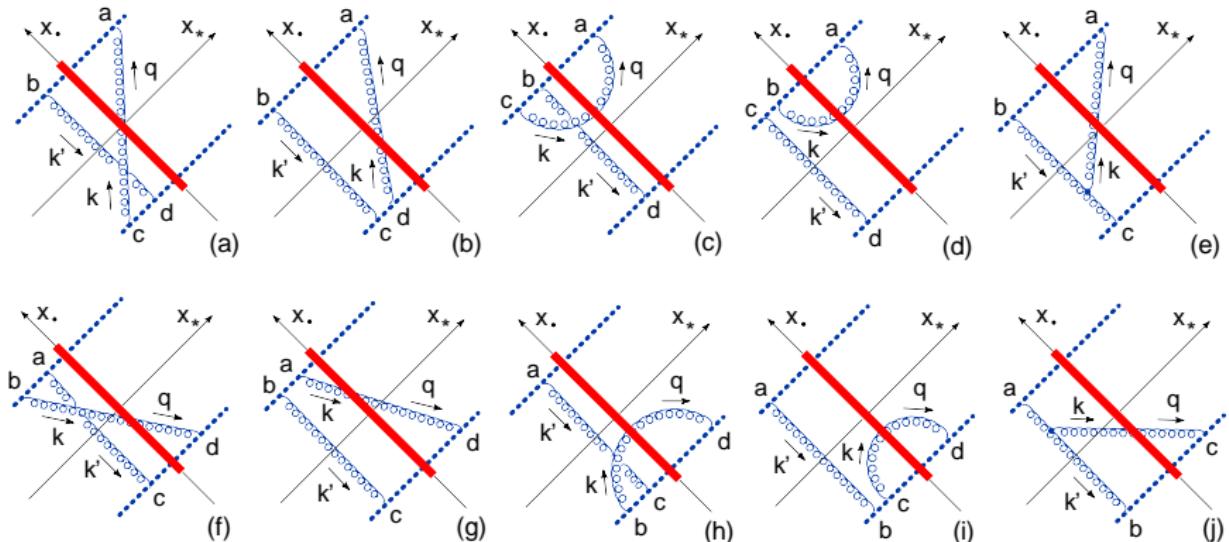
Diagrams of the NLO gluon contribution

"Running coupling" diagrams



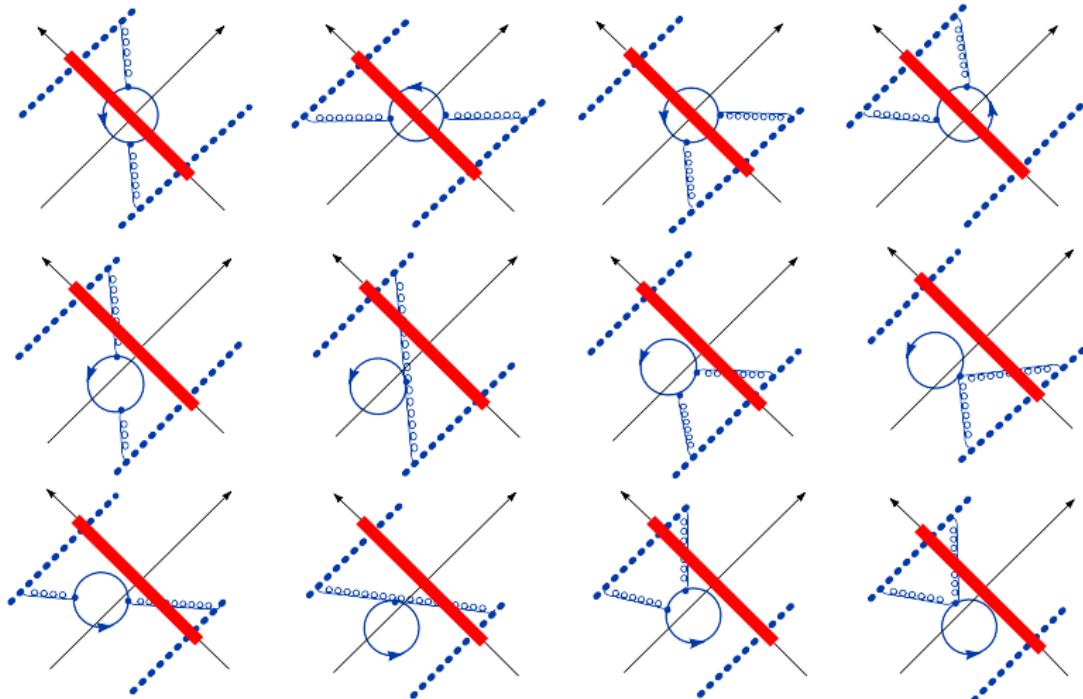
Diagrams of the NLO gluon contribution

1 → 2 dipole transition diagrams



Diagrams of the NLO gluon contribution

$\mathcal{N} = 4$ SYM diagrams (scalar and gluino loops)



$$\begin{aligned}
 \frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = & \frac{\alpha_s}{2\pi^2} \int d^2 z \left([\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
 & \times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[1 + \frac{\alpha_s N_c}{4\pi} \left(\frac{11}{3} \ln(x-y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \right. \\
 & - \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \Big\} \\
 & + \frac{\alpha_s}{4\pi^2} \int d^2 z' \left\{ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\
 & -(z' \rightarrow z)] \frac{1}{(z-z')^4} \left[-2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x-y)^2(z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \\
 & + [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z)] \\
 & \times \left. \left[\frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right\}
 \end{aligned}$$

Our result Agrees with NLO BFKL

(Comparing the eigenvalue of the forward kernel)

It respects unitarity

Gluon contribution to the NLO kernel (I. Balitsky and G.A.C, 2007)

$$\begin{aligned}
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2} \int d^2 z \left([\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\}] \right. \\
&\times \left\{ \frac{(x-y)^2}{X^2 Y^2} \left[1 + \frac{\alpha_s N_c}{4\pi} \left(\frac{11}{3} \ln(x-y)^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right) \right] \right. \\
&- \frac{11}{3} \frac{\alpha_s N_c}{4\pi} \frac{X^2 - Y^2}{X^2 Y^2} \ln \frac{X^2}{Y^2} - \frac{\alpha_s N_c}{2\pi} \frac{(x-y)^2}{X^2 Y^2} \ln \frac{X^2}{(x-y)^2} \ln \frac{Y^2}{(x-y)^2} \Big\} \\
&+ \frac{\alpha_s}{4\pi^2} \int d^2 z' \left\{ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_z^\dagger U_{z'} U_y^\dagger U_z U_{z'}^\dagger\} \right. \\
&\quad -(z' \rightarrow z)] \frac{1}{(z-z')^4} \left[-2 + \frac{X'^2 Y^2 + Y'^2 X^2 - 4(x-y)^2(z-z')^2}{2(X'^2 Y^2 - Y'^2 X^2)} \ln \frac{X'^2 Y^2}{Y'^2 X^2} \right] \\
&+ [\text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_{z'}^\dagger\} \{U_{z'} U_y^\dagger\} - \text{Tr}\{U_x U_{z'}^\dagger U_z U_y^\dagger U_{z'} U_z^\dagger\} - (z' \rightarrow z)] \\
&\quad \times \left. \left[\frac{(x-y)^4}{X^2 Y'^2 (X^2 Y'^2 - X'^2 Y^2)} + \frac{(x-y)^2}{(z-z')^2 X^2 Y'^2} \right] \ln \frac{X^2 Y'^2}{X'^2 Y^2} \right\}
\end{aligned}$$

NLO kernel = Running coupling terms + Non-conformal term + Conformal term

Evolution equation for color dipoles in $\mathcal{N} = 4$

(I. Balitsky and G.A.C. 2009)

$$\begin{aligned} & \frac{d}{d\eta} \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\ &= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ 1 - \frac{\alpha_s N_c}{4\pi} \left[\frac{\pi^2}{3} + 2 \ln \frac{z_{13}^2}{z_{12}^2} \ln \frac{z_{23}^2}{z_{12}^2} \right] \right\} \\ & \times [\text{Tr}\{T^a \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^a \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}] \\ & - \frac{\alpha_s^2}{4\pi^4} \int \frac{d^2 z_3 d^2 z_4}{z_{34}^4} \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{23}^2 z_{14}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \\ & \times \text{Tr}\{[T^a, T^b] \hat{U}_{z_1}^\eta T^{a'} T^{b'} \hat{U}_{z_2}^{\dagger\eta} + T^b T^a \hat{U}_{z_1}^\eta [T^{b'}, T^{a'}] \hat{U}_{z_2}^{\dagger\eta}\} (\hat{U}_{z_3}^\eta)^{ad'} (\hat{U}_{z_4}^\eta - \hat{U}_{z_3}^\eta)^{bb'} \end{aligned}$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha < \sigma = e^{2\eta}$ in the rapidity of Wilson lines.

Evolution equation for color dipoles in $\mathcal{N} = 4$

(I. Balitsky and G.A.C. 2009)

$$\begin{aligned} & \frac{d}{d\eta} \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\ &= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left\{ 1 - \frac{\alpha_s N_c}{4\pi} \left[\frac{\pi^2}{3} + 2 \ln \frac{z_{13}^2}{z_{12}^2} \ln \frac{z_{23}^2}{z_{12}^2} \right] \right\} \\ & \times [\text{Tr}\{T^a \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^a \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}] \\ & - \frac{\alpha_s^2}{4\pi^4} \int \frac{d^2 z_3 d^2 z_4}{z_{34}^4} \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{23}^2 z_{14}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \\ & \times \text{Tr}\{[T^a, T^b] \hat{U}_{z_1}^\eta T^{a'} T^{b'} \hat{U}_{z_2}^{\dagger\eta} + T^b T^a \hat{U}_{z_1}^\eta [T^{b'}, T^{a'}] \hat{U}_{z_2}^{\dagger\eta}\} (\hat{U}_{z_3}^\eta)^{ad'} (\hat{U}_{z_4}^\eta - \hat{U}_{z_3}^\eta)^{bb'} \end{aligned}$$

NLO kernel = Non-conformal term + Conformal term.

Non-conformal term is due to the non-invariant cutoff $\alpha < \sigma = e^{2\eta}$ in the rapidity of Wilson lines.

For the conformal composite dipole the result is Möbius invariant

Evolution equation for composite conformal dipoles in $\mathcal{N} = 4$ SYM

$$[\text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} = \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} + \frac{\alpha_s}{4\pi} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[\frac{1}{N_c} \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \right] \ln \frac{az_{12}^2}{z_{13}^2 z_{23}^2} + O(\alpha_s^2)$$

$$\begin{aligned} & \frac{d}{d\eta} [\text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} \\ &= \frac{\alpha_s}{\pi^2} \int d^2 z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[1 - \frac{\alpha_s N_c}{4\pi} \frac{\pi^2}{3} \right] [\text{Tr}\{T^a \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} T^a \hat{U}_{z_3} \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}]^{\text{conf}} \\ & - \frac{\alpha_s^2}{4\pi^4} \int d^2 z_3 d^2 z_4 \frac{z_{12}^2}{z_{13}^2 z_{24}^2 z_{34}^2} \left\{ 2 \ln \frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \left[1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right] \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right\} \\ & \times \text{Tr}\{[T^a, T^b] \hat{U}_{z_1}^\eta T^{a'} T^{b'} \hat{U}_{z_2}^{\dagger\eta} + T^b T^a \hat{U}_{z_1}^\eta [T^{b'}, T^{a'}] \hat{U}_{z_2}^{\dagger\eta}\} [(\hat{U}_{z_3}^\eta)^{aa'} (\hat{U}_{z_4}^\eta)^{bb'} - (z_4 \rightarrow z_3)] \end{aligned}$$

Now Möbius invariant!

NLO evolution of composite “conformal” dipoles in QCD

$$\begin{aligned}
 \frac{d}{d\eta} [\text{tr}\{\hat{U}_{z_1} \hat{U}_{z_2}^\dagger\}]^{\text{conf}} &= \frac{\alpha_s}{2\pi^2} \int d^2 z_3 \left([\text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_2}^\dagger\} - N_c \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_2}^\dagger\}]^{\text{conf}} \right. \\
 &\times \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[1 + \frac{\alpha_s N_c}{4\pi} (b \ln z_{12}^2 \mu^2 + b \frac{z_{13}^2 - z_{23}^2}{z_{13}^2 z_{23}^2} \ln \frac{z_{13}^2}{z_{23}^2} + \frac{67}{9} - \frac{\pi^2}{3}) \right] \\
 &+ \frac{\alpha_s}{4\pi^2} \int \frac{d^2 z_4}{z_{34}^4} \left\{ \left[-2 + \frac{z_{14}^2 z_{23}^2 + z_{24}^2 z_{13}^2 - 4 z_{12}^2 z_{34}^2}{2(z_{14}^2 z_{23}^2 - z_{24}^2 z_{13}^2)} \ln \frac{z_{14}^2 z_{23}^2}{z_{24}^2 z_{13}^2} \right] \right. \\
 &\times [\text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_4}^\dagger\} \{\hat{U}_{z_4} \hat{U}_{z_2}^\dagger\} - \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger \hat{U}_{z_4} \hat{U}_{z_2}^\dagger \hat{U}_{z_3} \hat{U}_{z_4}^\dagger\} - (z_4 \rightarrow z_3)] \\
 &+ \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[2 \ln \frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \left(1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right) \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right] \\
 &\times [\text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_4}^\dagger\} \text{tr}\{\hat{U}_{z_4} \hat{U}_{z_2}^\dagger\} - \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_4}^\dagger \hat{U}_{z_3} \hat{U}_{z_2}^\dagger \hat{U}_{z_4} \hat{U}_{z_3}^\dagger\} - (z_4 \rightarrow z_3)] \left. \right\}
 \end{aligned}$$

$$b = \frac{11}{3} N_c - \frac{2}{3} n_f$$

I. Balitsky and G.A.C

$K_{\text{NLO BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part
 + Conformal analytic ($\mathcal{N} = 4$) part

Linearized $K_{\text{NLO BK}}$ reproduces the known result for the forward NLO BFKL kernel Fadin and Lipatov (1998).

- High-energy operator expansion in color dipoles works at the NLO level.
- The analytic NLO photon impact factor in coordinate space has been calculated: the result is conformal.
- The NLO BK kernel in QCD and $\mathcal{N} = 4$ SYM agrees with NLO BFKL eigenvalues.
- The NLO BK kernel in QCD is a sum of the running-coupling part and conformal part.